RESIDENCE TIME DISTRIBUTION IN LAMINAR FLOW SYSTEMS. I. HYDRODYNAMIC CONCEPTION IN STUDY OF DISTRIBUTION FUNCTIONS

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On basis of the known velocity field in a steady laminar flow system can be determined the passing time along individual streamlines, identified by its end points in exit planes of the flow system. This primary information can be transformed into a form of residence time distribution functions which can be used in study of dynamic properties of considered flow systems such as chemical reactors or mixing devices. Hydrodynamically defined distribution functions, however, have in combination of several sub-systems into one entity different formal properties than analogous functions known from the classical theory of chemical flow reactors. This work presents the hydrodynamic definition of distribution functions and the study of some of their properties in forming combined models of laminar flow systems inclusive the questions of scaling-up and hydrodynamic similarities.

In flow of liquid through a flow system, the residence time of all liquid elements in the system is not the same. The simplest quantitative expression of this fact is given by some balance, for example by the distribution function of residence times $F(t)$ or by the density of distribution of residence times $E(t)$, for which is valid the relation

$$dt\, E(t) = F(t + dt) - F(t). \qquad (1)$$

The most common application of this integral description is the numerical estimate of the reaction conversion in the flow reactor with the known distribution density of residence times based on kinetic data on reaction obtained in a laboratory batch reactor[1,2]. The function of a flow system as a mixing equipment can be also described by distribution of residence times of the mixed materials[3,4]. Under certain assumptions, the distribution of residence times represents the ability of the flow system to smoothen the axial concentration gradients resp. the gradients of other quantities eventually transform them into gradients perpendicular to the flow direction[4,5]. Recently, characterization of the flow system by the residence time was used also in hydrodynamic studies of visco-elastic and rheotropic liquids[6], especially in solving the problems of their flow in porous media[7].

In systems, where the effect of dispersion processes is significant, *i.e.* in molecular or turbulent diffusions, the distribution functions have statistical character since they describe the result of stochastic processes[2]. In such case in their adequate application as well as in their evaluation from experiments of the "stimulus-response" type it is necessary to introduce further ideas and assumptions the most important of which concerns the question of identification of a particle according to its time of residence in the system and according to residual time of residence in the system (conception of macro- and microliquid[2], conception of the degree of segregation[8]) and the question of transport processes on the inlet and outlet surfaces (conception of closed and open systems[2], problematics of the so-called Danckwerts boundary conditions[9]).

In the case of laminar streamline flow through the flow system the distribution functions $F(t)$ and $E(t)$ can be defined exclusively on basis of the velocity field as its integral characteristics. Thus defined distribution functions do not have a statistical but fully deterministic balance nature. In relation with actual flow systems they are of course asymptotic approximations, as they describe exactly actual processes in flow systems only in case when the effect of molecular diffusion can be fully neglected. This assumption is in many technically interesting cases very well fulfilled, especially in operations with highly viscous liquids.

The usefullness of the mentioned approach seems to be also stressed by the experience concerning the difficulties in experimental determination of distribution functions in laminar flow systems with a little effect of dispersion processes on the overall mass transfer. If in the place of inducing a stimulus (usually be seeding the tracer) is not attained a good cross-mixing over the whole inlet area, the experimental results depend very much on the way of inducing the stimulus. If we feed for example the tracer too close to the wall into the region of stagnating laminar flow, we determine the residence times much longer then would correspond to reality. Analogous problems arise at insufficient cross-mixing at the exit from the system in the point where the response is measured. For instance the location of the probe for continuous determination of the analytical quantity can affect the result very much, among others it can give an impression of existence of dead regions in the flow system *etc.* In general it can be said that deviations of measured courses of distribution functions from the courses determined mathematically on the basis of known velocity fields can be almost always, in case of laminar flow systems with unimportant diffusiosion flows, ascribed to unsuitably performed experiments than to application of inadequate ideas in calculation of distribution functions.

The problem of calculation of distribution functions of residence times based on the known velocity fields and their application to the flow of highly viscous liquids has been very modestly studied so far. The only known works are concerned with the problematics of laminar flow in a tube both for Newtonian[10,11] and non-Newtonian liquids with the power-law viscosity function[12,13].

VELOCITY FIELDS AND DISTRIBUTION FUNCTIONS

Let us discuss the flow systems with steady laminar flow of an incompressible homogeneous liquid without volumetric sources and sinks. Velocity fields in the system are considered to be known. The whole volume in the system is the set of streamlines oriented in the direction of local velocities.

The boundary area of the system A_{tot} can be divided into three sub-classes: the area of the shell (resp. the wall) A_w which is formed by a system of streamlines; the inlet area A_0 which is intersected by all streamlines directed into the system; the outlet area A which is intersected by all streamlines directed outward from the system. As under the above given assumptions no streamline can end or begin in the system[14], to each of the points of the inlet area $\mathbf{x}_0 \in A_0$ can be ascribed only one outlet point of the area $\mathbf{x} \in A$ according to

$$\mathbf{x}_0 = \mathbf{p}(\mathbf{x}) \tag{2}$$

in a mutually single-valued way so that $\mathbf{x}$ is the end and $\mathbf{x}_0 = \mathbf{p}_0(\mathbf{x})$ the beginning of the same streamline in a considered system.

Further on we identify the individual streamlines by their end point $\mathbf{x}$ (*i.e.* by their intersection with the outlet area A) and determine their properties as a function of parameter $\mathbf{x} \in A$. We are interested especially in the passing time of a particle ϑ along the given streamline $\mathbf{x}$, for which holds

$$\vartheta = \vartheta(\mathbf{x}) \tag{3}$$

and in the outlet velocity vector $\mathbf{w}$ on the given streamline $\mathbf{x}$ for which

$$\mathbf{w} = \mathbf{w}(\mathbf{x}) \,. \tag{4}$$

We will demonstrate that the knowledge of functions (3) and (4) is sufficient for calculation of distribution functions E(t) and F(t).

DISTRIBUTION FUNCTIONS

Let us follow the liquid flowing from the considered system through the outlet area A. The oriented element of the outlet area $\mathrm{d}\mathbf{A}$ is chosen so as to be directed outward from the system. Elementary volumetric flow is expressed by the relation

$$\mathrm{d}Q(\mathbf{x}) = \mathbf{w}(\mathbf{x}) \,.\, \mathrm{d}\mathbf{A}(\mathbf{x}) \tag{5}$$

and the overall volumetric flow-rate is given by the integral over the outlet area according to

$$Q = \iint_{\mathbf{x} \in \mathbf{A}} \mathbf{w}(\mathbf{x}) \,.\, \mathrm{d}\mathbf{A}(\mathbf{x}) \,. \tag{6}$$

The passing time ϑ is a property of the liquid flowing from the system in a certain point of the outlet area $\mathbf{x} \in A$, resp. the property of this point of the outlet area. Based on this property, we can form sub-sets of points $\mathbf{x} \in A$ and make use of them in definition of distribution functions. Let $A_t \subset A$ be a set of that points of the outlet area through which flows out the liquid having the passing time ϑ equal or shorter than some given value t, then

$$\mathbf{x} \in A_t \Leftrightarrow (\mathbf{x} \in A \,\&\, \vartheta(\mathbf{x}) \leqq t)\,. \tag{7}$$

The distribution function of residence times $\mathrm{F}(t)$ is according to definition the ratio of liquid flowing through the area A_t to the total amount of flowing liquid, resp. the ratio of corresponding flow rates, and is given by

$$\mathrm{F}(t) = 1/Q \iint_{\mathbf{x} \in A_t} \mathbf{w}(\mathbf{x}) \,.\, \mathrm{d}\mathbf{A}(\mathbf{x})\,. \tag{8}$$

The distribution density of residence times $\mathrm{E}(t)$ could be in principle determined by derivation of the function $\mathrm{F}(t)$. This procedure which is rather complex and leading to inaccuracies in actual cases can be avoided and $\mathrm{E}(t)$ function can be calculated directly from the known functions (*3*) and (*4*).

Let us begin with the definition of *E*-function in Eq. (*1*). This relation can be, after substituting for $\mathrm{F}(t)$ according to Eq. (*8*), written in the form

$$Q\,\mathrm{d}t\,\mathrm{E}(t) = \iint_{A_{t+\mathrm{d}t}} \mathbf{w} \,.\, \mathrm{d}\mathbf{A} - \iint_{A_t} \mathbf{w} \,.\, \mathrm{d}\mathbf{A}\,, \tag{9a}$$

or

$$Q\,\mathrm{d}t\,\mathrm{E}(t) = \iint_{A_{t,\mathrm{d}t}} \mathbf{w} \,.\, \mathrm{d}\mathbf{A}\,, \tag{9b}$$

where $A_{t,\mathrm{d}t}$ is an elementary area defined by relation

$$\mathbf{x} \in A_{t,\mathrm{d}t} \Leftrightarrow (\mathbf{x} \in A \,\&\, \vartheta(\mathbf{x}) \in (t, t + \mathrm{d}t\rangle) \tag{10}$$

i.e. as the set of points of the outlet area through which passes the liquid with the residence time ϑ falling into the interval $(t, t + \mathrm{d}t\rangle$, where t and $\mathrm{d}t$ are quantities chosen independently.

The field of passing times $\vartheta(\mathbf{x})$ is a continuous and continuously differentiable function so that the area $A_{t,\mathrm{d}t}$ is limited by two systems of curves K_t and $K_{t,\mathrm{d}t}$ of final length which are sets of end-points of streamlines $\mathbf{x} \in A$ on which the residence time is just equal to t resp. $t + \mathrm{d}t$ (Fig. 1):

$$\mathbf{x} \in K_t \Leftrightarrow (\mathbf{x} \in A \,\&\, \vartheta(\mathbf{x}) = t)\,, \tag{11a}$$

$$\mathbf{x} \in K_{t+\mathrm{d}t} \Leftrightarrow (\mathbf{x} \in A \,\&\, \vartheta(\mathbf{x}) = t + \mathrm{d}t)\,. \tag{11b}$$

In an arbitrarily chosen point $\mathbf{x} \in K_t$ there can be fixed a local orthonormal base $\mathbf{e}_A$, $\mathbf{e}_K$, $\mathbf{e}_N$ (Fig. 2) while $\mathbf{e}_K$ and $\mathbf{e}_N$ are situated in a plane tangential to the outlet flow area at point $\mathbf{x} \in K$. The surface gradient[15] of the function $\vartheta(\mathbf{x})$ can be written in local Cartesian coordinates as

$$\nabla\vartheta = \frac{\partial\vartheta}{\partial x_A}\,\mathbf{e}_A + \frac{\partial\vartheta}{\partial x_K}\,\mathbf{e}_K = \frac{\partial\vartheta}{\partial x_N}\,\mathbf{e}_N = \frac{\partial\vartheta}{\partial x_N}\,\mathbf{e}_N\,, \qquad (12)$$

since ϑ is not a function of x_A, and x_K is tangential to the curve K_t of the constant $\vartheta(\mathbf{x}) = t$ at a given point $\mathbf{x}$. For the given difference $\mathrm{d}t$ between the curves K_t and $K_{t+\mathrm{d}t}$ is the distance between these curves $\mathrm{d}x_N(x)$ variable in dependence on the local value of the gradient $\nabla\vartheta$, according to

$$\mathrm{d}x_N(\mathbf{x}) = \frac{\mathrm{d}t}{|\nabla\vartheta(\mathbf{x})|}\,. \qquad (13)$$

The differential of area $\mathrm{d}A(x)$ at a given place $\mathbf{x} \in K_t$ (Fig. 2) can be, therefore, written in the form

$$\mathrm{d}\boldsymbol{A}_{t,\mathrm{d}t} = \mathrm{d}x_N\,\mathrm{d}x_K\mathbf{e}_A = \mathrm{d}t\,\frac{\mathrm{d}K\mathbf{e}_A(\mathbf{x})}{|\nabla\vartheta(\mathbf{x})|}\,, \qquad (14)$$

as for the differentials holds the equality $\mathrm{d}K = \mathrm{d}x_K$. By substitution into Eq. (*9a*) for $\mathrm{d}A$ according to Eq. (*14*) we obtain the relation

$$Q\,\mathrm{d}t\,\mathrm{E}(t) = \int_{K_t} \mathrm{d}t\,\frac{\mathbf{w}(\mathbf{x})\,.\,\mathbf{e}_A(\mathbf{x})}{|\nabla\vartheta(\mathbf{x})|}\,\mathrm{d}K \qquad (15a)$$

and thus

$$\mathrm{E}(t) = 1/Q \int_{K_t} \frac{w_A(\mathbf{x})}{|\nabla\vartheta(\mathbf{x})|}\,\mathrm{d}K\,, \qquad (15b)$$

where

$$w_A(\mathbf{x}) = \mathbf{w}(\mathbf{x})\,.\,\mathbf{e}_A(\mathbf{x})\,. \qquad (16)$$

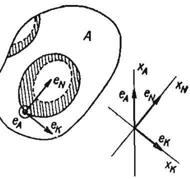

FIG. 1

Mapping of the Outlet Area $\mathbf{x} \in A$ on Basis of Surface Field of Passing Times $\vartheta(\mathbf{x})$

Hatched area $A_{t,\mathrm{d}t}$, which in general is not continuous, is limited by the system of curves K_t (solid lines) and $K_{t+\mathrm{d}t}$ (dashed lines). On the curve K_t is the origin of the orthogonal base, where $\mathbf{e}_A$ has the direction $\mathrm{d}A$, $\mathbf{e}_K$ is tangential to the curve K_t and $\mathbf{e}_N$ has the direction of gradient of function $\vartheta(\mathbf{x})$.

Flow Kinematics

The kinematics of a steady flow can be described in principle in two ways — in the Euler way, *i.e.* by giving the velocity fields $\mathbf{v}$ in the Euclidean system $\mathbf{r}$ of the external observer,

$$\mathbf{v} = \mathbf{v}(\mathbf{r})\,; \quad \mathbf{r} \in V\,, \tag{17}$$

or in the Lagrange way, *i.e.* by giving the hodograph (dependence of the instant position $\mathbf{r}$ on the suitable scalar parameter, most often the time variable) of a arbitrary particle identified by its position $\mathbf{r}_0$ at a chosen time $t = t_0$, which is

$$\mathbf{r} = \mathbf{p}(t - t_0, \mathbf{r}_0)\,, \tag{18a}$$

$$\mathbf{r}_0 = \mathbf{p}(0, \mathbf{r}_0)\,. \tag{18b}$$

The result of solution of a hydrodynamic problem is usually the Eulerian description of kinematics (*17*). We are interested in transforming this description into data on the passing time ϑ of particles and on the inlet point $\mathbf{x}_0$ of particles into the system through given area A_0 in dependence on the point $\mathbf{x}$ of their outlet from the system through the given area A, *i.e.* in determining the course of functions (*3*) and (*4*).

It we consider the local velocity $\mathbf{v}(\mathbf{r})$ of the liquid as a velocity of motion of some identified particles of the liquid, the relation holds

$$\mathrm{d}\mathbf{p}/\mathrm{d}t = \mathbf{v}(\mathbf{p})\,, \tag{19a}$$

where $\mathbf{p}$ is the instant position of the considered particle and t the time measured by the observer. When we measure the time for the given particle from its entry into the system through the point $\mathbf{x}_0$, *i.e.* if we introduce the new time variable τ' by

$$\tau' = t - t_0\,, \tag{20}$$

where $t = t_0$ is the moment of entrance of the particle into the system, the relation (*19a*) can be then written as

$$\mathrm{d}\mathbf{p}/\mathrm{d}\tau' = \mathbf{v}(\mathbf{p}) \tag{19b}$$

with initial conditions

$$\mathbf{p} = \mathbf{x}_0\,; \quad \tau' = 0\,. \tag{21}$$

The integral of differential system (*19b*) and (*21*) are relations

$$\mathbf{p} = \mathbf{p}(\tau')\,, \tag{22a}$$

which satisfy the condition

$$\mathbf{x}_0 = \mathbf{p}(0)\,. \tag{22b}$$

The relations *(22a)* are meaningful only for $\tau' \in (0; \tau'_0)$, where τ'_0 is the maximum value of the parameter τ', satisfying the relation $\boldsymbol{p}(\tau'_0) \in A$.

Under certain conditions, which are always satisfied for steady laminar flow of an incompressible liquid in a system without volumetric sources and sinks, there exists for each $\boldsymbol{x}_0 \in A_0$ just one $\boldsymbol{x} \in A$ and thus only one $\tau'_0 \in (0; \infty)$. It is thus possible, by integration of the systems *(19a)* and *(21)* to obtain hodographs*

$$\boldsymbol{p} = \boldsymbol{p}'(\tau', \boldsymbol{x}_0) \quad \boldsymbol{x}_0 \in A_0 , \quad \tau' \in (0, \tau'_0) , \tag{23}$$

while at the same time for each $\boldsymbol{x}_0 \in A_0$ exists only one τ'_0 (the passing time for the streamline $\boldsymbol{x}_0$) given by

$$\tau'_0 = \tau'_0(\boldsymbol{x}_0) , \tag{24}$$

satisfying the relation

$$\boldsymbol{x} = \boldsymbol{p}'(\tau'_0(\boldsymbol{x}_0), \boldsymbol{x}_0) = \boldsymbol{p}'_0(\boldsymbol{x}_0) , \tag{25}$$

by which there exists between $\boldsymbol{x} \in A$ and $\boldsymbol{x}_0 \in A_0$ a single-valued correspondence.

As the function *(25)* is invertible, it holds

$$\boldsymbol{x}_0 = \boldsymbol{p}_0(\boldsymbol{x}) , \quad \boldsymbol{x} \in A, \boldsymbol{x}_0 \in A_0 \tag{26}$$

and the passing time along the given streamline $\boldsymbol{x}$ resp. $\boldsymbol{x}_0 = \boldsymbol{p}_0(\boldsymbol{x})$ can be also expressed as a function of parameter $\boldsymbol{x}$, according to

$$\vartheta = \vartheta(\boldsymbol{x}) = \tau'_0(\boldsymbol{p}_0(\boldsymbol{x})) . \tag{27}$$

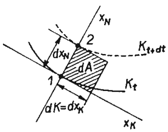

FIG. 2

Differential of Area $d(A_{t,dt})$

Point *1* is located at $\boldsymbol{x}$, point *2* at $\boldsymbol{x} + \boldsymbol{e}_N \, dx_N$.

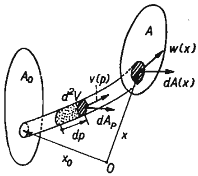

FIG. 3

Flow through the Elementary Stream Tube

A_0 Inlet area, A outlet area, 0 origin of the reference coordinate system.

* Relation *(23)* is as well the parametric expression of the streamline, passing through the point $\boldsymbol{x}_0$.

An alternate procedure giving the same results is introduction of parameter τ as

$$\tau = t_x - t\,, \tag{28}$$

where t is the time and $t = t_x$ is the instant of exit of the particle from the system in the point $\mathbf{x} \in A$, and integration of the corresponding differential system

$$-\mathrm{d}\mathbf{p}/\mathrm{d}\tau = \mathbf{v}(\mathbf{p})\,; \quad \mathbf{p} \in V\,. \tag{19c}$$

with the initial condition

$$\mathbf{p} = \mathbf{x}\,; \quad \tau = 0\,. \tag{29}$$

The result is the hodographic function in the form

$$\mathbf{p} = \mathbf{p}(\tau, \mathbf{x}) \tag{30}$$

from which the required informations can be obtained in the form

$$\mathbf{p}(\tau, \mathbf{x}) \in A_0\,, \quad \tau = \vartheta(\mathbf{x})\,; \tag{31}$$

$$\mathbf{p}(\vartheta(\mathbf{x}), \mathbf{x}) \equiv \mathbf{p}_0(\mathbf{x})\,. \tag{32}$$

Mean Residence Time

Let us consider the passing time ϑ to be an intensive property of the liquid leaving the system in the point $\mathbf{x}$, in a sense that to the mixture of volumes V_j of the liquid with the passing time $\vartheta_j j = 1, \ldots n$ can be ascribed the mean value $\bar{\vartheta}$ according to the mixing rule

$$\bar{\vartheta} = \frac{\sum \vartheta_j V_j}{\sum V_j}\,, \tag{33a}$$

resp., in case of mixing of different streams of invariable quality with constant flow rates Q_j according to

$$\bar{\vartheta} = \frac{\sum \vartheta_j Q_j}{\sum Q_j}\,. \tag{33b}$$

In laminar flow systems, according to the given definition, the mean residence time can be obviously expressed as

$$\bar{\vartheta} = \frac{\iint_A \vartheta(\mathbf{x})\,.\,\mathrm{d}Q(\mathbf{x})}{\iint_A \mathrm{d}Q(\mathbf{x})} = \frac{1}{Q}\iint_A \vartheta(\mathbf{x})\,.\,\mathbf{w}(\mathbf{x})\,.\,\mathrm{d}\mathbf{A}(\mathbf{x})\,. \tag{34}$$

Let us consider the elementary stream tube across the whole flow system, limiting in the inlet area $\mathbf{x} \in A$ an element $\mathrm{d}\mathbf{A}(\mathbf{x})$, see Fig. 3. Let us limit a reference area $\mathrm{d}\mathbf{A}_p$ across the stream tube in its arbitrary position $\mathbf{p} \in V$, $\mathbf{p} = \mathbf{p}(\mathbf{x}, \tau)$, $\tau \in (0, \vartheta(\mathbf{x}))$ so that the stream tube would be completely closed by it. According to the continuity equation the volumetric flow rate of incompressible liquid is constant in randomly chosen cross-section of the stream tube, *i.e.* const $= \mathbf{v}(\mathbf{p}(\mathbf{x}, \tau)) \,.\, \mathrm{d}\mathbf{A}_p(\mathbf{p}(\mathbf{x}, \tau)) = \mathbf{w}(\mathbf{x}) \,.\, \mathrm{d}\mathbf{A}(\mathbf{x})$. According to relation *(19c)* the elementary length of the stream tube $\mathrm{d}\mathbf{p}$ can be expressed as $\mathrm{d}\mathbf{p} = -\mathbf{v}(\mathbf{p}) \,.\, \mathrm{d}\tau$ and the corresponding elementary volume of the stream tube d^2V (Fig. 3) can be expressed as $\mathrm{d}^2V = \mathrm{d}\mathbf{A}_p(\mathbf{p}) \,.\, \mathrm{d}\mathbf{p} = (\mathrm{d}\mathbf{A}_p(\mathbf{p}) \,.\, \mathbf{v}(\mathbf{p})\, \mathrm{d}\tau = \mathbf{w}(\mathbf{x}) \,.\, \mathrm{d}\mathbf{A}(\mathbf{x})\, \mathrm{d}\tau$. The overall volume of the elementary stream tube is thus given by the relation

$$\mathrm{d}V = \mathbf{w}(\mathbf{x}) \,.\, \mathrm{d}\mathbf{A}(\mathbf{x}) \int_0^{\vartheta(\mathbf{x})} \mathrm{d}\tau \qquad (35a)$$

and the volume of the flow system as a sum of volumes of all stream tubes is consequently given by relation

$$V = \iint_A \vartheta(\mathbf{x}) \,.\, \mathbf{w}(\mathbf{x}) \,.\, \mathrm{d}\mathbf{A}(\mathbf{x}) \,. \qquad (35b)$$

From comparison of relations *(34)* and *(35)* follows

$$\bar{\vartheta} = V/Q \,. \qquad (36)$$

The same conclusion can be obtained in a simple way if we imagine individual stream tubes as elementary piston flow systems* with passing time ϑ_j, volumes V_j^0 and flow rates Q_j, $j = 1, \ldots, n$. For a flow system with piston flow holds[2]

$$\vartheta_j = \bar{\vartheta}_j = V_j^0/Q_j \,.$$

According to definition of the mean residence time for steady flow systems *(33b)* it holds

$$\bar{\vartheta} = \frac{\sum (V_j^0/Q_j)\, Q_j}{\sum Q_j} = \frac{\sum V_j^0}{\sum Q_j} = \frac{V}{Q} \,,$$

where $V = \sum V_j^0$ is the sum of volumes of all stream tubes, leading into the outlet area A.

Normalization of Distribution Functions and Kinematic Similarity

According to definition of the mean residence time *(33b)* and of function $\mathrm{E}(t)$ by Eqs *(9b)*, *(10)* holds

* Flow system with piston flow is a system in which all particles of the liquid have equal residence time[2].

$$\bar{t} = \int_0^\infty t\,\mathrm{E}(t)\,\mathrm{d}t = \bar{\vartheta} = V/Q\,. \tag{37}$$

The mean residence time $\bar{t}$ can be used for normalization of $\mathrm{E}(t)$ function. The normalized time variable is defined by relation

$$\Theta = t/\bar{t} \tag{38}$$

and E function is normalized by parameter t according to

$$\mathrm{E}^+(\Theta) = \bar{t}\,\mathrm{dF}(t)/\mathrm{d}t = \bar{t}\,\mathrm{E}(\Theta\bar{t})\,; \tag{39}$$

so that the normalized E^+ function complies with the two normalizing conditions

$$\int_0^\infty \mathrm{E}^+(\Theta)\,\mathrm{d}\Theta = 1\,. \tag{40a}$$

$$\int_0^\infty \Theta\,\mathrm{E}^+(\Theta)\,\mathrm{d}\Theta = 1\,. \tag{40b}$$

Of the two flow systems which have identical $\mathrm{E}^+(\Theta)$ function we speak as of the systems with similar residence time distribution. The question of distribution similarity is significant most of all in applications where it is the theoretical basis for evaluation of theoretical correlations, but it is also useful in our further studies as it is related to a great extent with the question of kinematic similarity of flow systems.

Two flow systems are kinematicaly similar if their velocity fields with the use of dimensionless quantities

$$\mathbf{v}^* = \mathbf{v}/U\,, \tag{41a}$$

$$\mathbf{r}^* = \mathbf{r}/R\,, \tag{41b}$$

can be written in an identical form

$$\mathbf{v}^*(\mathbf{r}^*) = \text{idem}\,. \tag{42}$$

As a consequence of normalization *(41b)* of the Euclidean reference system and of normalization of the velocity fields *(41a)* exists a number of relations defining dimensionless equivalent quantities which we use. To geometrical points of curves, areas and volumes $\mathbf{r} \in K, A, V$, now correspond relations $\mathbf{r}^* \in K^*, A^*, V^*$ and the metrics of normalized forms is given by relations for their differentials, which are

$$\mathrm{d}\boldsymbol{K}^* = \mathrm{R}^{-1}\,\mathrm{d}\boldsymbol{K}\,, \tag{43a}$$

$$\mathrm{d}\boldsymbol{A}^* = \mathrm{R}^{-2}\,\mathrm{d}\boldsymbol{A}\,, \tag{43b}$$

$$\mathrm{d}\boldsymbol{V}^* = \mathrm{R}^{-3}\,\mathrm{d}\boldsymbol{V}\,. \tag{43c}$$

Quantities $\boldsymbol{x}_0$ and $\boldsymbol{x}$ and $\boldsymbol{p}$ are eléments of certain sub-sets V and therefore they are normalized in the same way as radiusvector $\boldsymbol{r}$. The spacial and surface nabla-operator are normalized identically as

$$\nabla^* = R\nabla\,. \tag{43d}$$

The integral balances (6) and (35) can be written as

$$Q^* = \frac{Q}{UR^2} = \iint_{A^*} \boldsymbol{w}^*(\boldsymbol{x}^*)\,.\,\mathrm{d}\boldsymbol{A}^*(\boldsymbol{x}^*)\,, \tag{44a}$$

$$V^* = \frac{V}{R^3} = \frac{V}{\bar{t}\,UR^2}\iint_{A^*} \vartheta^+(\boldsymbol{x}^*)\,.\,\boldsymbol{w}^*(\boldsymbol{x}^*)\,.\,\mathrm{d}\boldsymbol{A}^*(\boldsymbol{x}^*)\,, \tag{44b}$$

where, according to definition (38), we introduce the dimensionless equivalent of the passing time by relation

$$\vartheta^+(\boldsymbol{x}^*) = \vartheta(R\boldsymbol{x}^*)/\bar{\vartheta}\,, \tag{45}$$

where, according to relation (37), is $\bar{t} = \bar{\vartheta} = V/Q$.

Expression of the course of normalized $\mathrm{E}^*(\Theta)$ function with the use of normalized description of the flow kinematics can be now obtained by simple substitution according to relations (38), (39), (43), (44) into the relation (15b)

$$\mathrm{E}^+(\Theta) = 1/Q^* \int_{K_{\Theta^*}} \frac{w_A^*(\boldsymbol{x}^*)\,.\,\mathrm{d}K^*(\boldsymbol{x}^*)}{|\nabla^*\vartheta^+(\boldsymbol{x}^*)|}\,, \tag{46}$$

where

$$\boldsymbol{x}^* \in K_0^* \Leftrightarrow \vartheta^+(\boldsymbol{x}^*) = \Theta\,, \tag{47}$$

and $\vartheta^+(\boldsymbol{x}^*)$ is defined by relation (45).

Normalized passing times given by relation (45) are usually obtained in solving a specified problem not from the data in dimensional form (31), but as a result of solution of a dimensionless hydrodynamic problem where standardized dimensional factors U and R are chosen independently on the alternative pair of kinematic normalizing factors Q and V. In such case it is usual to define the dimensionless time variable in some other way, usually as

$$t^* = tU/R\,, \tag{48}$$

because only then no numerical factors appear in the normalized differential equation (19c) and in the corresponding hodographic function *(30)* which are thus given by

$$-\mathrm{d}\mathbf{p}^*/\mathrm{d}t^* = \mathbf{v}^*(\mathbf{p}^*)\,, \tag{49}$$

$$\mathbf{p}^* = \mathbf{p}^*(t^*, \mathbf{x}^*) = (1/R)\,\mathbf{p}(t^*R/U, \mathbf{x}^*R)\,. \tag{50}$$

The corresponding data on the passing time have obviously the form

$$t^* = t^*(\mathbf{x}^*) \Leftrightarrow \mathbf{p}^*(t^*(\mathbf{x}^*), \mathbf{x}^*) \in A_0^*\,, \tag{51}$$

which can be transformed into relation *(47)*:

$$\vartheta^+ = t^*Q^*/V^*\,, \tag{52}$$

where Q^* and V^*, defined by relations *(44a,b)* give the transformation relations between independent pairs of normalizing factors U and R resp. V and Q.

Since, however, for kinematically similar systems according to relation *(44a, b)* is Q^* = idem, V^* = idem, and according to relations *(49)*, *(50)*, *(51)* also $t^*(\mathbf{x}^*)$ = idem, obviously according to Eq. *(46)* is for them as well $\mathrm{E}^+(\Theta)$ = idem. The kinematic similarity is not an indispensable condition for similarity of residence time distributions because different velocity fields can provide the same integral result, *e.g.*, a parallel set of identical flow systems has obviously the same normalized distribution functions as one flow system of this set taken as one unit, though there can be no discussion on a kinematic similarity in the sense used here. Sufficient condition for similarity of distribution functions is the kinematic similarity only in the case considered by us when the effect of dispersion processes on mass transfer can be neglected.

Combination of Streamline Systems

The studied flow systems are very often effectively divided into two or more sub-systems for which the velocity fields have specific symmetries or simple analytical expression and only then the results obtained for these sub-systems are used in the terms of residence time distributions. In the case of streamline flow come into consideration two basical ways of distribution: into parallel sub-systems which are formed by independent stream tubes (Fig. 4), or into sub-systems in a series formed by consecutive sections of the same stream tube (Fig. 5). These two basical forms can be, of course, combined in various ways similarly like in the theory of flow

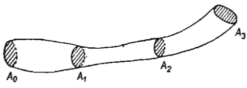

Fig. 4

Parallel Combination of Streamline Flow Systems, $n = 3$

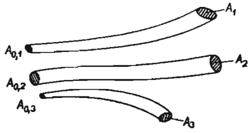

Fig. 5

Serial Combination of Streamline Systems, $m = 3$

reactors[2] models of real systems are formed by series and parallel combining of two basical classical flow elements of an ideal flow mixer and an ideal piston flow system.

The question which is for us of the prime interest in respect to combination of the streamline systems is the extent of informations on individual sub-systems necessary for their synthesis into description of distribution functions of the system as a whole.

Parallel Combination

The F-function is defined as the ratio of liquid from the total amount which has a certain property t, *i.e.* the residence time lessthan or equal to t. If the total amount of liquid passing through the i-th sub-system Q_i, the quantity $Q_i\, F_i(t)$ in the i-th sub-system has the mentioned property. In a system consisting of n sub-systems is the total amount of the passed liquid with quality t given by the sum of amounts in individual parallel flows, and thus it holds

$$F(t) = \sum_{i=1}^{n} a_i\, F_i(t)\,, \tag{53}$$

where

$$a_i = Q_i/Q\,, \tag{54}$$

$$Q = \sum_{i=1}^{n} Q_i\,. \tag{55}$$

$F(t)$ is the distribution function of the system as a whole and Q is the volumetric flow-rate through the system as a whole. According to definition of the E-function holds as well

$$E(t) = \sum_{i=1}^{n} a_i\, E_i(t)\,. \tag{56}$$

The normalized F-function differs from the non-normalized one only by the argument because F-function itself is normalized by the condition $F(\infty) = 1$. It is therefore possible to write the definition of a normalized function F^+ as

$$F^+(\Theta) = F(\bar{t}\Theta)\,. \tag{57}$$

Let us begin with the assumption that for individual sub-systems we know their normalized distribution functions F_i^+ and E_i^+, their volumes V_i and flow-rates Q_i, resp. the mean residence times $\bar{t}_i = V_i/Q_i$. The mean residence time of the system as a whole is according to definition $\bar{t} = V/Q$, and the corresponding normalized time variable is $Q = tQ/V$. Dimensionless equivalents of relations (56) and (57) then have the form

$$F^+(\Theta) = a_i\, F_i^+(\Theta/b_i)\,, \tag{58}$$

$$E^+(\Theta) = (a_i/b_i)\, E_i^+(\Theta/b_i)\,, \tag{59}$$

where

$$b_i = \bar{t}_i/t = (V_i/Q_i)\,(Q/V)\,. \tag{60}$$

Serial Combination in Streamline Connection of the Sub-Systems

Let the system as a whole be formed by a single stream tube divided into m sections (sub-systems) by control areas $A_1 \ldots, A_{m-1}$ and terminated by the outlet area A_m (Fig. 5). We assume that each of the sub-systems is characterized by functions $\vartheta_i(\boldsymbol{x}_i)$ of residence times, and $\boldsymbol{x}_{i-1} = \boldsymbol{x}_{0,i} = \boldsymbol{p}_{0,i}(\boldsymbol{x}_i)$ of inlet points with the argument $\boldsymbol{x}_i \in A_i$ the local exit point. Volumetric flow-rate Q is for all sub-systems the same different are their volumes V_i and their mean residence time $\bar{t}_i = V_i/Q$.

Each streamline can be identified by its intersection with any of control areas A_i, $i = 1, \ldots, m$. Let us choose one of them as a reference area of the whole system, for example A_m. On the basis of known function

$$\boldsymbol{x}_{i-1} = \boldsymbol{p}_{0,i}(\boldsymbol{x}_i)\,, \tag{61}$$

sets of mutual correspondence can ben be formed of individual intersections of the same streamline $\boldsymbol{x}_0, \boldsymbol{x}_i, \ldots, \boldsymbol{x}_m$ with control areas of the sub-systems. This partial correspondence are thus single-valued and thus invertable so that on their basis may be formed mutually single-valued correspondence of points of any two control areas A_i, A_j. For the chosen control area A_m the correspondence may be formed by a composite function without inversions, which is

$$\boldsymbol{x}_i = \boldsymbol{p}_{0,i+1}(\ldots \boldsymbol{p}_{m-1}(\boldsymbol{x}_m) \ldots) = \boldsymbol{p}_{i,m}(\boldsymbol{x}_m)\,. \tag{62}$$

Thus to point $\boldsymbol{x}_m$ correspond also the passing times $\vartheta_i(\boldsymbol{x}_i)$ in individual sections of a given streamline according to

$$\vartheta_i(\boldsymbol{x}_i) = \vartheta_i(\boldsymbol{p}_{i,m}(\boldsymbol{x}_m) = \vartheta_{i,m}(\boldsymbol{x}_m)\,. \tag{63}$$

The total passing time along the streamline is the sum of passing times along its individual sections

$$\vartheta(\boldsymbol{x}_m) = \sum_{i=1}^{m} \vartheta_{i,m}(\boldsymbol{x}_m)\,, \tag{64}$$

so that the gradient in the area A_m of function $\vartheta(\boldsymbol{x}_m)$ is the sum of gradients of functions (*64*); to their formation (*63*) is, however, necessary to know also the correspondence of individual control areas (*62*) for which

$$\nabla_m\, \vartheta(\boldsymbol{x}_m) = \sum_{i=1}^{m} \nabla_m\, \vartheta_{i,m}(\boldsymbol{x}_m)\,. \tag{65}$$

The E(t) function of the system as a whole can be now written by Eq. (*15b*) as

$$\mathrm{E}(t) = 1/Q \int_{K_t} \frac{w_A(\mathbf{x}_m)\,\mathrm{d}K}{\sum_{i=1}^{m} \left|\nabla_m \vartheta_{i,m}(\mathbf{x}_m)\right|}, \tag{66}$$

where

$$\mathbf{x}_m \in K_t \Leftrightarrow t = \sum_{i=1}^{m} \vartheta_{i,m}(\mathbf{x}_m) \tag{67}$$

and where $\mathbf{x}_m \in A_m$ and ∇_m is the surface differential operator[15] in the area A_m.

Now, the normalized description of the flow kinematics in individual sections is considered to be the primary information, at the assumption, that normalizing factors R and U and Q as well are for all sub-systems equally chosen. Factors V_i are for individual systems in general different. According to assumption is $\mathbf{x}_i^* = \mathbf{x}_i/R$, $i = 1, \ldots, m$ so that functions (*63*) are normalized trivially

$$\mathbf{x}_i^* = \mathbf{p}_{i,m}^*(x_k^*) .$$

The passing times are normalized by the local mean times $\bar{t}_i = V_i/Q$, so that the normalized passing time ϑ^+ through the system as a whole is given by relation

$$\vartheta^+(\mathbf{x}_m^*) = (Q/V)\,\vartheta(\mathbf{x}_m^* R) = \sum_{i=1}^{m} c_i\, \vartheta_{i,m}^+(\mathbf{x}_m^*), \tag{68}$$

where $V = V_i$, $c_i = V_i/V$ so that according to Eq. (*66*) is

$$\mathrm{E}^+(\Theta) = (V/Q)\,\mathrm{E}(\Theta V/Q) = 1/Q^* \int_{K^*_\Theta} \frac{w_A^*(\mathbf{x}_m^*)\,\mathrm{d}K^*}{\sum_{i=1}^{m} \left|c_i \nabla_m^* \vartheta_{i,m}^+(\mathbf{x}_m^*)\right|}, \tag{69}$$

where

$$\mathbf{x}_m^* \in K_0^* \Leftrightarrow \Theta = \sum_{i=1}^{m} c_i\, \vartheta_{i,m}^+(\mathbf{x}_m^*) . \tag{70}$$

After arrangement into a series it holds $\sum_{i=1}^{m} t_i = \sum_{i=1}^{m} V_i/Q = V/Q = \bar{t}$.

CONCLUSION

In classical works dealing with the questions of residence time distribution in agitated flow systems[1,2,8,9] it is without mentioning assumed to a certain extent perfect mixing of liquid across the inlet and outlet areas of the flow system so that any distribution functions of outlet streams and consequently the distribution function of residence times have a naturel of concentration data valid for a sample of liquid taken in any point of the outlet area. This assumption is not, however, fulfilled in the flow of highly viscous liquids, characterized overwhelmingly by laminar, streamline character of the flow and by very low coefficients of molecular diffusion. Over the inlet and outlet areas there exist in such case very expressive gradients of quantities characterizing the properties of liquid and the resulting effect, for inst. reaction in a cascade of mixed vessels, depends very strongly on the flow pattern at the place of their connection. Actual situation can, therefore, in certain respects often approach the hypothetic situation considered in this work, *i.e.* the situation when the only transport is the laminar convection.

As it is obvious from considerations on combination of streamline systems, in such case servev for an adequate description of the situation the field of passing times $\vartheta(\mathbf{x})$ and of outlet velocities $\mathbf{w}(\mathbf{x})$, from which the distribution functions can be quite easily determined with the use of relations (*15b*), (*11a*) resp. (*8*), (*7*). The only information on distribution functions is not specific enough and on the other hand data on complete velocity fields are too wide for discussion of properties of flow system such as a reactor or a mixer.

As follows from the characteristics of most of the highly viscous liquids, characterized often by non-Newtonian flow anomalies, it is not realistic to base the hydrodynamic approach to the problematics of laminar flow systems on the dynamics of Newtonian isothermal flow. Therefore we have made no assumption on properties of the velocity field from those which follow from requirements of its continuity and from the continuity equation for liquids with constant density.

LIST OF SYMBOLS

a_i	parameters, see Eq. (*54*)
A	outlet area
A_0	inlet area
A_{tot}	total boundary area
A_w	area of the shell or of the walls
A_t	part of outlet area, see Eq. (*7*)
$A_{t,dt}$	part of outlet area, see Eq. (*10*)
b_i	parameter, see Eq. (*60*)
c_i	parameter, see Eq. (*68*)
$\mathbf{e}_i$	vector of orthogonal basis
$E(t)$	density of residence time distribution
$E^+(\Theta)$	normalized distribution density
$F(t)$	residence time distribution
$F^+(\Theta)$	normalized distribution
K_t	intersection of the outlet area and streamlines of constant passing time
$\mathbf{p}$	possition of liquid particle, hodograph
$\mathbf{p}_0(\mathbf{x})$	mutual correspondence of inlet and outlet point situated on the same streamline
$\mathbf{r}$	radiusvector in Euclidean space

R	characteristic dimension of the system, normalizing factor
t	time recorded by the observer, residence time in the system
$\mathbf{v}$	local velocity
$\mathbf{w}(\mathbf{x})$	local velocity in the outlet area
$\mathbf{x}, \mathbf{x}_0, \mathbf{x}_w$	points of planes A, A_0, A_w respectively
$w_A(\mathbf{x})$	projection of w into the normal line to A in point $\mathbf{x}$
$\vartheta(\mathbf{x})$	passing time along the streamline with the end point $\mathbf{x}$
τ, τ'	mean residence time of the particle in the system, see Eq. (*20*) and (*28*)
$\tau_0'(x_0)$	passing time along the streamline with initial point $\mathbf{x}_0$
—	quantities averaged according to flow rate
*	dimensionless quantities normalized by parameter U and R
+	dimensionless quantities, normalized by parameter V and Q

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